L.
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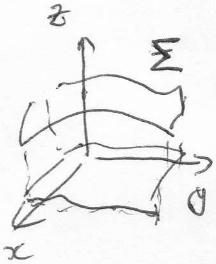
Minimal surfaces in $\mathbb{R}^3$. Enneper-Weierstrass formula



$z = u(x,y)$

$$g_{|\Sigma} = dx^2 + dy^2 + dz^2 =$$

$$= (1+u_x^2)dx^2 + 2u_x u_y \, dx\,dy + (1+u_y^2)dy^2$$

metric tensor
$$g_{\mu\nu} = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix}$$

$$\sqrt{\det g_{\mu\nu}} = \sqrt{1+u_x^2+u_y^2}$$

$$S[u] = \int_u \sqrt{1+u_x^2+u_y^2} \, dx \wedge dy$$

$$\delta S = 0 \implies \Sigma \text{ is minimal.}$$

Euler-Lagrange equations: $\mathcal{L} = \sqrt{1+u_x^2+u_y^2}$

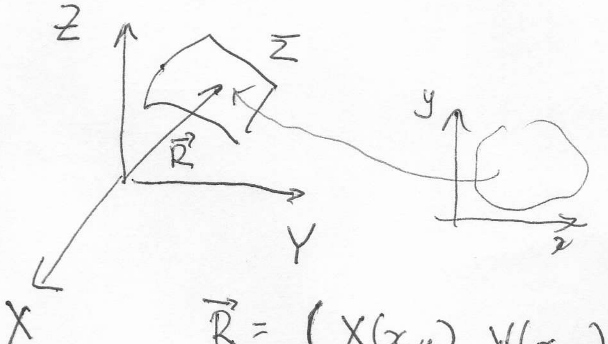
$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\implies \frac{\partial}{\partial x}\frac{u_x}{\sqrt{\quad}} + \frac{\partial}{\partial y}\frac{u_y}{\sqrt{\quad}} = 0$$

$$\rightsquigarrow \boxed{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0} \quad *$$

Easy to see that $u = ax+by+c$ satisfies $*$, but ~~it~~ is difficult to find other solutions.

Another approach Weierstrass ~ 1865



$$\vec{R} = \left(X(x,y), Y(x,y), Z(x,y) \right)$$

e.g. $X = x$, $Y = y$, $Z = u(x,y)$.

$$g = d\vec{R}^2 \; ; \quad g|_\Sigma = \left(\vec{R}_x dx + \vec{R}_y dy\right)^2 =$$
$$= \vec{R}_x^2 dx^2 + 2\vec{R}_x \vec{R}_y dx\, dy + \vec{R}_y^2 dy^2.$$

$$\det g = \vec{R}_x^2 \cdot \vec{R}_y^2 - \left(\vec{R}_x \cdot \vec{R}_y\right)^2 = \left(\vec{R}_x \times \vec{R}_y\right)^2 = L^2$$

minimal condition:

$$\delta \int L \, dx \wedge dy = 0$$

$$\Rightarrow \frac{\partial}{\partial x} \frac{\partial L}{\partial \vec{R}_x} + \frac{\partial}{\partial y} \frac{\partial L}{\partial \vec{R}_y} = 0$$

w.r.t. δR_x

$$2\mathcal{L}\,\delta\mathcal{L} = \delta\mathcal{L}^2 = \delta\left(\vec{R}_x\times\vec{R}_y\right)^2 =$$

$$= 2\left(\delta\vec{R}_x\times\vec{R}_y\right)\cdot\left(\vec{R}_x\times\vec{R}_y\right)$$

$$= 2\left(\vec{R}_y\times\left(\vec{R}_x\times\vec{R}_y\right)\right)\delta\vec{R}_x$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \left(\vec{b}\times\vec{c}\right)\cdot\vec{a}$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_x} = \frac{1}{\mathcal{L}}\left(\vec{R}_y\times\left(\vec{R}_x\times\vec{R}_y\right)\right)$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_y} = \frac{1}{\mathcal{L}}\left(\vec{R}_x\times\left(\vec{R}_y\times\vec{R}_x\right)\right)$$

From the very begining I could have restricted my attention to coordinates (x,y) in which

$$g|_\Sigma = e^{2\varphi}\left(dx^2+dy^2\right)$$ i.e. to the coordinates in which $\vec{R}_x^{\,2} = \vec{R}_y^{\,2}$ and $\vec{R}_x\cdot\vec{R}_y = 0$.

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_x} = \vec{R}_x \quad , \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_y} = \vec{R}_y$$

$$\Rightarrow \quad \boxed{\vec{R}_{xx} + \vec{R}_{yy} = 0}$$

$$+ \begin{cases} \vec{R}_x^{\,2} = \vec{R}_y^{\,2} \\ \vec{R}_x\cdot\vec{R}_y = 0 \end{cases}$$

$$\Rightarrow \boxed{\vec{R} = \operatorname{Re} \vec{F}(z)}$$

$\uparrow$ holomorphic in $z = x + iy$

$$+ \text{ conditions } \begin{cases} \vec{R}_x^{\,2} = \vec{R}_y^{\,2} \\ \vec{R}_x \cdot \vec{R}_y = 0 \end{cases}$$

$$\vec{F} = \vec{R} + i\vec{S}$$

$$\vec{F}'(z) = \frac{d\vec{F}}{dz} = \frac{1}{2}(\partial_x - i\partial_y)\vec{F} =$$

$$= \frac{1}{2}\left(\vec{F}_x - i\vec{F}_y\right) = \frac{1}{2}\left(\vec{R}_x + i\vec{S}_x - i\vec{R}_y + \vec{S}_y\right)$$

$$\underset{\uparrow \text{ Cauchy-Riemann}}{=\!=\!=} \frac{1}{2}\left(\vec{R}_x - i\vec{R}_y - i\vec{R}_y + \vec{R}_x\right) =$$

$$\begin{cases} \vec{R}_x = \vec{S}_y \\ \vec{R}_y = -\vec{S}_x \end{cases} \qquad = \vec{R}_x - i\vec{R}_y$$

$$\vec{F}' \cdot \vec{F}' = \left(\vec{R}_x - i\vec{R}_y\right)^2 = \vec{R}_x^{\,2} - \vec{R}_y^{\,2} - 2i\vec{R}_x\vec{R}_y = 0$$

$\vec{F}'$ must be a holomorphic complex NULL vector in $\mathbb{C}^3$

$$\vec{F}' = (X, Y, Z)$$

$$A = \begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix} \Rightarrow \det A = -Z^2 - X^2 - Y^2 = -\vec{F}'^2 = 0$$

$$\Rightarrow A = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}(\psi_1, \psi_2) = \begin{pmatrix} \varphi_1\psi_1 & \varphi_1\psi_2 \\ \varphi_2\psi_1 & \varphi_2\psi_2 \end{pmatrix}$$

$$\begin{cases} Z = \varphi_1 \psi_1 = -\varphi_2 \psi_2 \\ X - iY = \varphi_1 \psi_2 \\ X + iY = \varphi_2 \psi_1 \end{cases} \Rightarrow \psi_2 = -\frac{\varphi_1}{\varphi_2}\psi_1$$

$$X - iY = -\frac{\varphi_1^2}{\varphi_2}\psi_1$$

$$\frac{\psi_1}{\varphi_2} = \alpha$$

$$X + iY = \varphi_2 \psi_1$$

$$\left.\begin{array}{l} \begin{cases} X - iY = -\varphi_1^2 \alpha \\ X + iY = \varphi_2^2 \alpha \end{cases} \\ Z = \varphi_1 \varphi_2 \cdot \alpha \\ \varphi = \frac{\varphi_1}{\sqrt{\alpha}}, \quad \psi = \frac{\varphi_2}{\sqrt{\alpha}} \end{array}\right\} \Rightarrow \begin{cases} Z = \varphi\psi \\ X - iY = -\varphi^2 \\ X + iY = \psi^2 \end{cases}$$

$$\boxed{\begin{cases} X = \dfrac{\psi^2 - \varphi^2}{2} \\ Y = \dfrac{-\varphi^2 - \psi^2}{2i} \\ Z = \varphi\psi \end{cases}}$$

$\Rightarrow$

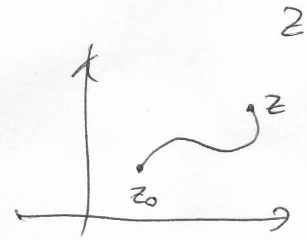
$$\begin{cases} \cancel{Z = \varphi\psi} \\ \cancel{X = -\varphi^2} \\ \cancel{Y = \psi^2} \end{cases}$$

$$\frac{\psi^4 - 2\psi^2\varphi^2 + \varphi^4}{4} - \frac{\varphi^4 + 2\psi^2\varphi^2 + \psi^4}{4} + \psi^2\varphi^2 = 0$$

ok

$$\Rightarrow \quad \vec{F}'(z) = \begin{pmatrix} \dfrac{\varphi^2 - \psi^2}{2} \\ \dfrac{\varphi^2 + \psi^2}{2i} \\ \varphi\psi \end{pmatrix}$$

where
$\psi = \psi(z)$
$\varphi = \varphi(z)$ holomorphic



$$\Rightarrow \quad \vec{F}(z) = \int \begin{pmatrix} \dfrac{\varphi^2 - \psi^2}{2} \\ \dfrac{\varphi^2 + \psi^2}{2i} \\ \varphi\psi \end{pmatrix} dz'$$

integration along any curve starting at $z' = z_0$ to $z' = z$
variable.

$\vec{R} = \mathrm{Re}\,\vec{F}$.

— . —

minimal surfaces are special case of harmonic maps.

$$(M, h) \xrightarrow{\varphi} (N, g)$$

$$(x^\mu, h_{\mu\nu}) \qquad (y^A, g_{AB})$$

$$\cancel{\Delta_h \varphi^A + \Gamma^A_{BC}\, \varphi^B \varphi^C = 0}$$

$\cancel{g}$

$$\boxed{\Delta_h \varphi^A + \Gamma^A_{BC}\, \varphi^B_{,\mu}\, \varphi^C_{,\nu}\, h^{\mu\nu} = 0}$$

$M \subset \mathbb{R}^1$
$\Downarrow$
$\varphi(M)$ is a geodesic.